ZÁVEREČNÝ TEST ZS 2022/2023 Variant D

1.

$$\lim_{n\to\infty} \frac{3^{2n+2}+16^{n+1}}{7^{n+1}-4^{2n+1}+16^{-n}} = \lim_{n\to\infty} \frac{(3^2)^{n+1}+16\cdot 16^n}{7\cdot 7^n - 4\cdot(4^2)^n + 16^{-n}} = \lim_{n\to\infty} \frac{9\cdot 9^n + 16\cdot 16^n}{7\cdot 7^n - 4\cdot 16^n + 16^{-n}} =$$

$$= \lim_{n\to\infty} \frac{16^n\left(9\cdot\left(\frac{9}{16}\right)^n + 16\right)}{16^n\left(7\cdot\left(\frac{7}{16}\right)^n - 4 + 16^{-2n}\right)} = \lim_{n\to\infty} \frac{9\cdot\left(\frac{9}{16}\right)^n + 16}{7\cdot\left(\frac{7}{16}\right)^n - 4 + 16^{-2n}} = -\frac{16}{4} = -4$$

($\left(\frac{9}{16}\right)^n \to 0$, $7\cdot\left(\frac{7}{16}\right)^n \to 0$, $16^{-2n} \to 0$)

- správna úprava na rozumný tvar pred vytknutím 1.5b
- vytknutie správnych členov a správna úprava 1.5 b
- vyhodnotenie výrazov idúcich k nule a správny výsledok 1b

2. $f(x) = (2x^2+3)\cdot \ln(2x+4)$

D_f: $2x+4>0$
$x>-2 \Rightarrow D_f = (-2,\infty)$ 0.75b

$f'(x) = (2x^2+3)'\cdot \ln(2x+4) + (2x^2+3)\cdot(\ln(2x+4))' =$ 1b

$= 4x\cdot\ln(2x+4) + (2x^2+3)\cdot \frac{1}{2x+4}\cdot 2 =$ 1b (where $2 = (2x+4)'$)

$= 4x\cdot\ln(2x+4) + \frac{2x^2+3}{x+2}$ 0.5b

$D_{f'}$: $x+2\neq 0 \wedge 2x+4>0$
$x\neq -2 \wedge x>-2$

$D_{f'} = D_f = (-2,\infty)$ 0.75 b

3. $f(x) = 2x^2+6x-8$ $\quad y = kx+q \quad k=2$

$f'(x) = 4x+6$ 1b

$2 = 4x_0+6$ 1b

$-4 = 4x_0$

$x_0 = -1$ 0.5b

T: $y_0 = 2\cdot(-1)^2+6\cdot(-1)-8 = 2-6-8 = -12$

$T = [-1,-12]$ 1b

q: $-12 = 2\cdot(-1)+q$
$-12 = -2+q$

$q = -10$ 1b

$y = 2x-10$ 0.5b

parabola:

P_x: $2(x^2+3x-4) = 0$

$x_1+x_2 = -3$, $x_1\cdot x_2 = -4 \Rightarrow x_1 = -4$, $x_2 = 1 \Rightarrow [-4,0]$, $[1,0]$ 1.5b

P_y: $y = 2\cdot 0^2+6\cdot 0-8 = -8 \Rightarrow [0,-8]$ 0.5b

vrchol: $x_v = \frac{x_1+x_2}{2} = \frac{-4+1}{2} = -\frac{3}{2}$

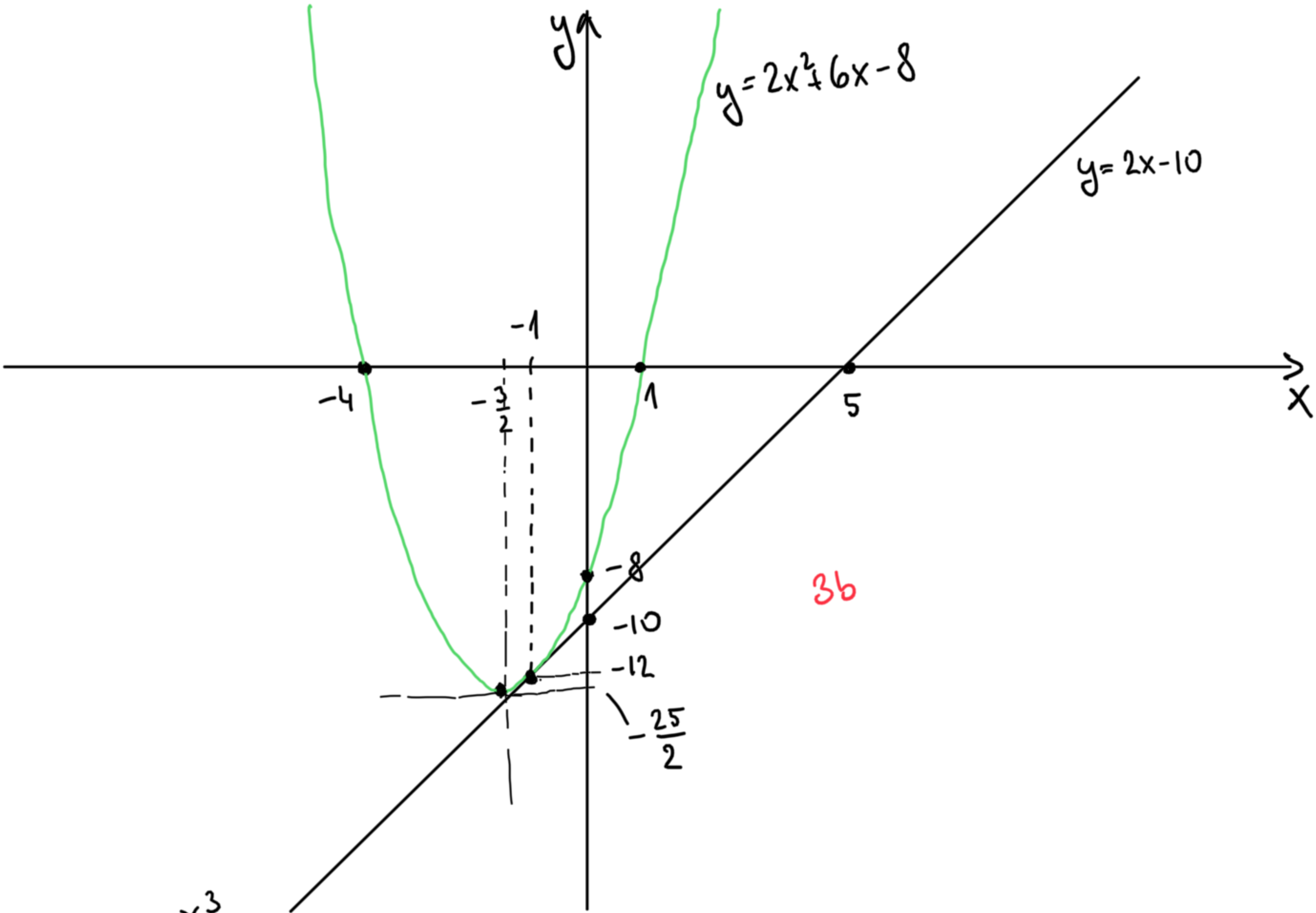
tečna:

P_x: $2x+10=0$, $x=-5 \Rightarrow [5,0]$ 0.5b

P_y: $y = 2\cdot 0-10 = -10 \Rightarrow [0,-10]$ 0.5b

$\Rightarrow y_v = 2\left(\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) - 4\right) = 2\cdot\left(\frac{9}{4} - \frac{9}{2} - \frac{16}{4}\right)$

$= 2\left(\frac{9}{4} - \frac{18}{4} - \frac{16}{4}\right) = -\frac{25}{2} \quad \Rightarrow \quad \boxed{V = \left[-\frac{3}{2}, -\frac{25}{2}\right]}$ 1b



3b

④ $f(x) = \frac{x^3}{x^2-1}$

1. $D_f: x^2 - 1 \neq 0 \Rightarrow x \neq \pm 1$ $\boxed{D_f = \mathbb{R} \setminus \{\pm 1\}}$ (sym. def. obor) 1b

2. $f(-x) = \frac{(-x)^3}{(-x)^2 - 1} = \frac{-x^3}{x^2-1} = -f(x) \Rightarrow$ **funkcia je lichá** 1b

3. $P_y: \; y = \frac{0^3}{0^2-1} = 0$

   $P_x: \; 0 = \frac{x^3}{x^2-1} \Rightarrow x = 0$

   $P_y: [0,0]$
   $P_x: [0,0]$ 1b

funkciu stačí celý čas vyšetrovať na polovici D_f napr. $\langle 0, \infty) \setminus \{1\}$ a správne symetricky dokresliť

4. SB: $\{-1, 0, 1\}$ 0,5b

	$(-\infty,-1)$	$(-1,0\rangle$	$\langle 0,1)$	$(1,\infty)$
x^3	−	−	+	+
x^2-1	+	−	−	+
$f(x)$	⊖	⊕	⊖	⊕

fcia je:
kladná na $(-1,0\rangle \cup (1,\infty)$
záporná na $(-\infty,-1) \cup \langle 0,1)$ 1b

5. $\lim\limits_{x\to\pm\infty} \frac{x^3}{x^2-1} = \lim\limits_{x\to\pm\infty} \frac{x^3}{x^2\left(1-\frac{1}{x^2}\right)} = \lim\limits_{x\to\pm\infty} \frac{x}{1-\frac{1}{x^2}} = \underline{\pm\infty}$ 0,5b

$$\lim_{x\to 1^+} \frac{x^3}{x^2-1} = \frac{1}{0^+} = +\infty \quad 0{,}5b \qquad \lim_{x\to -1^+} \frac{x^3}{x^2-1} = \frac{-1}{0^-} = +\infty \quad 0{,}5b$$

$$\lim_{x\to 1^-} \frac{x^3}{x^2-1} = \frac{1}{0^-} = -\infty \quad 0{,}5b \qquad \lim_{x\to -1^-} \frac{x^3}{x^2-1} = \frac{-1}{0^+} = -\infty \quad 0.5b$$

6. $f(x) = \dfrac{x^3}{x^2-1}$

$$f'(x) = \frac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2} = \boxed{\frac{x^2(x^2-3)}{(x^2-1)^2}} \quad 1b$$

NB: $\{-\sqrt{3}, -1, 0, 1, \sqrt{3}\}$ 0,5b

7.

	$(-\infty, -\sqrt{3}\rangle$	$\langle -\sqrt{3}, -1)$	$(-1, 0\rangle$	$\langle 0, 1)$	$(1, \sqrt{3}\rangle$	$\langle \sqrt{3}, \infty)$
x^2	+	+	+	+	+	+
x^2-3	+	−	−	−	−	+
$(x^2-1)^2$	+	+	+	+	+	+
$f'(x)$	⊕	⊖	⊖	⊖	⊖	⊕
$f(x)$	↗	↘	↘	↘	↘	↗

→ 1b

8.

$$f(\sqrt{3}) = \frac{3\cdot\sqrt{3}}{3-1} = \frac{3}{2}\cdot\sqrt{3} \doteq 2.6 \qquad \boxed{\text{lok. max } [\sqrt{3}, 2.6]} \text{ resp. } [\sqrt{3}, \tfrac{2}{3}\sqrt{3}]$$

$$f(-\sqrt{3}) = \frac{-3\sqrt{3}}{3-1} = -\frac{3}{2}\cdot\sqrt{3} \doteq -2.6 \qquad \boxed{\text{lok min } [-\sqrt{3}, -2.6]} \text{ resp. } [-\sqrt{3}, -\tfrac{2}{3}\sqrt{3}]$$

1b

9. $\boxed{H_f = \mathbb{R}}$ 1b

10. $\boxed{\text{dve svislé asymptoty } x = \pm 1}$ 0,5b

šikme:

$$k = \lim_{x\to\pm\infty} \frac{\frac{x^3}{x^2-1}}{x} = \lim_{x\to\pm\infty} \frac{x^3}{x(x^2-1)} = \lim_{x\to\pm\infty} \frac{1}{1-\frac{1}{x^2}} = 1$$

$$q = \lim_{x\to\pm\infty} \frac{x^3}{x^2-1} - x = \lim_{x\to\pm\infty} \frac{x^3 - x^3 + x}{x^2-1} = \lim_{x\to\pm\infty} \frac{1}{x(1+\frac{1}{x})} = 0$$

šikmá asymptota

$y = x$ 1b

11. $f'(x) = \dfrac{x^4 - 3x^2}{(x^2-1)^2}$

$$f''(x) = \frac{(4x^3 - 6x)(x^2-1)^2 - (x^4 - 3x^2)\,2\,(x^2-1)\cdot 2x}{(x^2-1)^4} =$$

$$= \frac{(4x^3 - 6x)(x^2-1) - (x^4 - 3x^2)\cdot 4x}{(x^2-1)^3} =$$

$$= \frac{4x^5 - 4x^3 - 6x^3 + 6x - 4x^5 + 12x^3}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

1b

NB: $0, \pm\sqrt{3}, \pm 1$ 0,5b

	$(-\infty,-\sqrt{3}\rangle$	$\langle-\sqrt{3},-1)$	$(-1,0\rangle$	$\langle 0,1)$	$(1,\sqrt{3}\rangle$	$\langle\sqrt{3},\infty)$
$2x$	−	−	−	+	+	+
x^2+3	+	+	+	+	+	+
$(x^2-1)^3$	+	+	−	−	+	+
$f''(x)$	⊖	⊖	⊕	⊖	⊕	⊕
$f(x)$	∩	∩	∪	∩	∪	∪

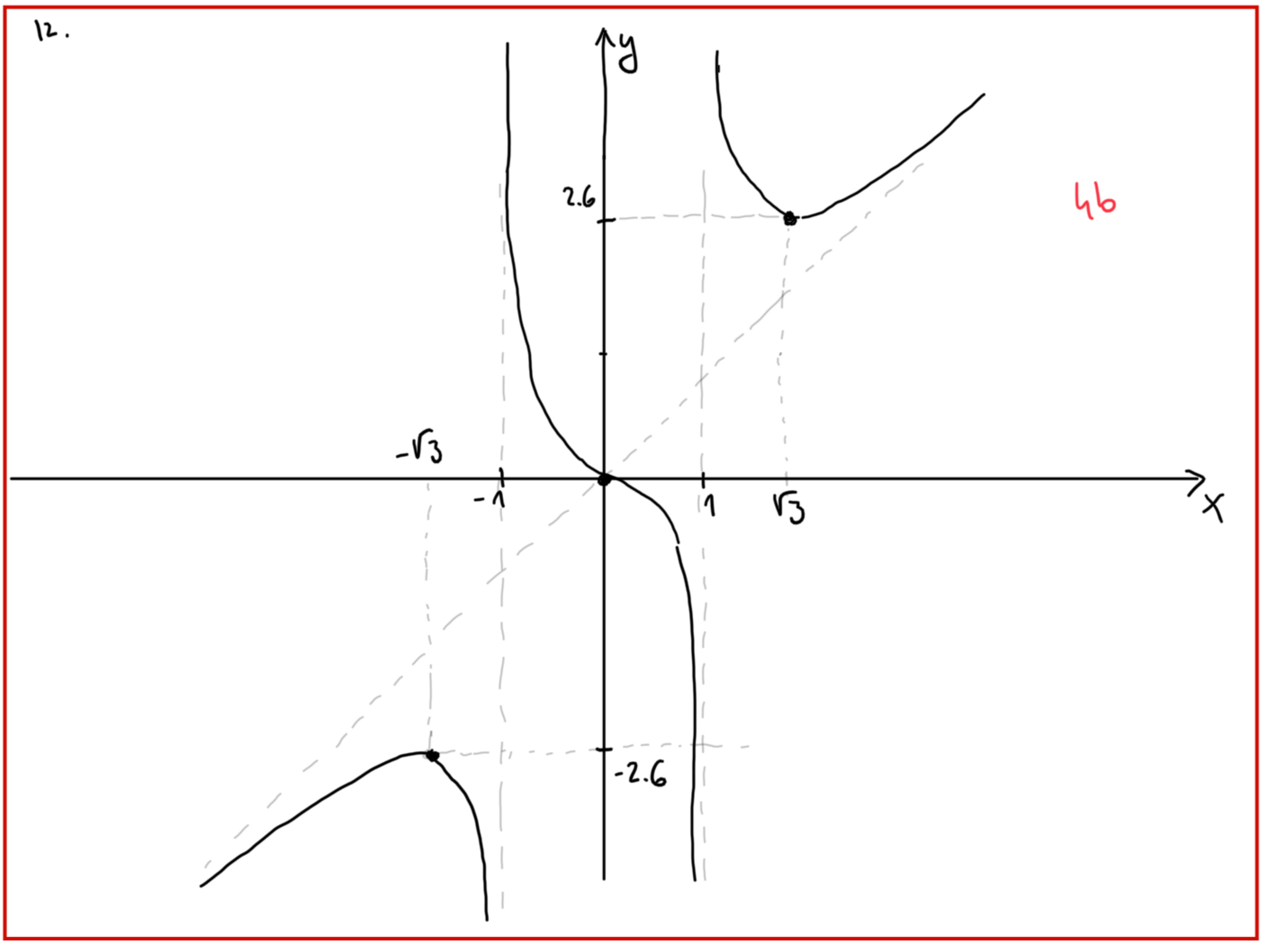
funkcia je konvexná na intervaloch

$(-1,0\rangle \cup (1,\infty)$ 1b

=> inflexný bod $[0,0]$ 0.5b

funkcia je konkávna na intervaloch

$(-\infty,-1) \cup \langle 0,1)$



5.

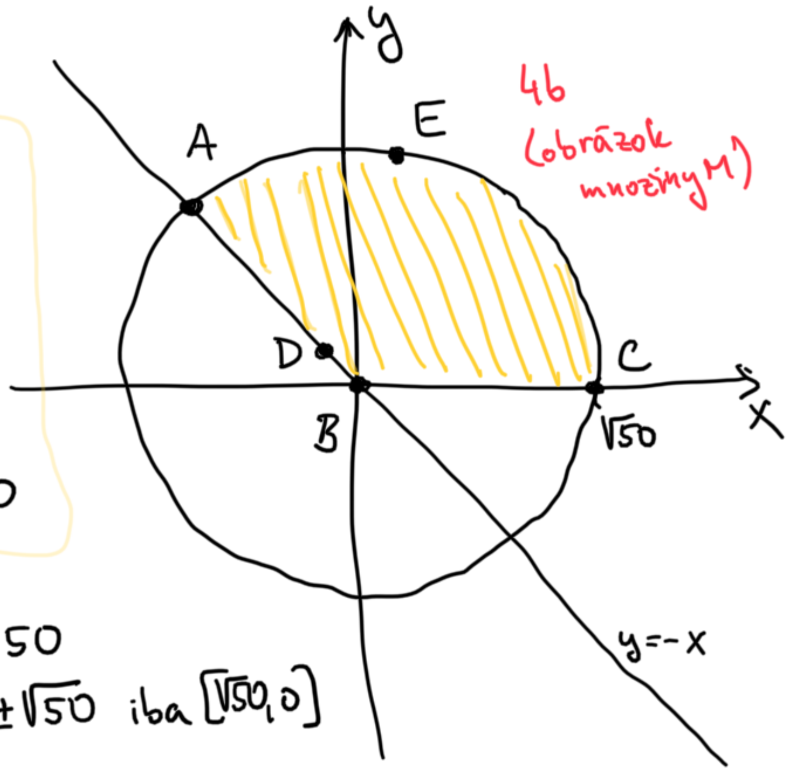
$f(x,y) = 2x + y^2 \quad M = \{[x,y] \in \mathbb{R} : y \geq 0,\ x+y \geq 0,\ x^2+y^2 \leq 50\}$

$\sqrt{50} \doteq 7.07$

MNOŽINA M

- $x^2+y^2 \leq 50$ -> kruh s polomerom $\doteq 7.07$
- $x+y \geq 0$

  $y \geq -x$ polrovina oddelená priamkou $y=x$
- $y \geq 0$ polrovina oddelená priamkou $y=0$



4b (obrázok množiny M)

- $x^2+x^2=50$

  $2x^2=50$

  $x^2=25 \Rightarrow x=\pm 5 \quad y=\pm 5$

  $\Rightarrow$ iba $[-5,5]$

  $[5,-5] \notin M$
- $x^2+0=50$

  $x=\pm\sqrt{50}$ iba $[\sqrt{50},0]$
- $-x=0 \Rightarrow [0,0]$

$A=[5,-5]$ 1b
$B=[0,0]$ 0.5b
$C=[\sqrt{50},0]$ 1b

M^o

$\partial_x f(x,y) = 2 \neq 0$ 1b

$\partial_y f(x,y) = 2y$ 1b $\Rightarrow$ žiaden kandidát vo vnútri množiny 0,5b

∂M priamka AB $\quad x \in \langle -5,5 \rangle \quad y=-x$

$f(x,-x) = 2x+(-x)^2 = 2x+x^2$ 1b

$f'(x,-x) = 2+2x = 0 \Leftrightarrow x=-1$ 0.5b

$y=1$

bod **D $[-1,1]$** 1b

∂M priamka BC $\quad x \in \langle 0,5 \rangle \quad y=0$

$f(x,0) = 2x$ 1b

$f' = 2 \neq 0$ 0.5b $\Rightarrow$ na tomto úseku nemáme kandidáta na extrém 0.5b

∂M kruhový oblúk

$f(x,y,\lambda) = 2x+y^2+\lambda(x^2+y^2-50)$ 0,5b

$\partial_x f = 2 + 2\lambda x = 0$ 0.5b

$\partial_y f = 2y + 2\lambda y = 0$ 0.5b $\Leftrightarrow$ $y = 0$ 0.5b $\vee$ $\lambda = -1$ 0.5b

$\partial_\lambda f = x^2 + y^2 - 50 = 0$ 0.5b

- $y = 0 \Rightarrow x^2 - 50 = 0$ $\quad\quad 1 + \lambda \cdot \sqrt{50} = 0$

  $x^2 = 50$

  $x = \pm\sqrt{50}$ iba $\sqrt{50}$ ($-\sqrt{50}$ nepatrí ∂M ani M°)

kandidáta $[\sqrt{50}, 0]$ už máme (bod C) 0.5b

- $\lambda = -1 \Rightarrow 2 + 2 \cdot (-1) x = 0 \Rightarrow x = 1$ 0.5b

  $1 - 50 + y^2 = 0$

  $y^2 = 49$

  $y = \pm 7$ 0.5b

pre $\lambda = -1$ dostávame kandidáta $E = [1, 7]$

kandidáti: $f(x,y) = 2x + y^2$

A $[-5, 5]$	$\rightarrow$ $10 + 25 = 35$		
B $[0, 0]$	$\rightarrow$ $0 + 0 = 0$		
C $[\sqrt{50}, 0]$	$\rightarrow$ $2 \cdot \sqrt{50} \doteq 14.14$		
D $[-1, 1]$	$\rightarrow$ $-2 + 1 = -1$	MIN	
E $[1, 7]$	$\rightarrow$ $2 + 49 = 51$	MAX	

2b